JOURNAL OF COMBINATORIAL THEORY, Series A **77**, 349–356 (1997)
ARTICLE NO. TA972737

NOTE

Another Deformation of Weyl's Denominator Formula

Todd Simpson

Department of Mathematics, Pennsylvania State University, University Park, Pennsylvania 16802

Communicated by the Managing Editors

Received January 24, 1995

INTRODUCTION

Our purpose is to prove an identity much like Okada's [O] deformations of Weyl's denominator formula. This identity will specialize to Weyl's formula for root systems of type B. The reader should consult [O] as needed for definitions and notations.

Given a positive integer n, we denote by $\langle n \rangle$ the set $\{1, 2, ..., n\}$. A sequence $(s_1, s_2, ..., s_k)$ of 1's, 0's, and -1's is said to be sign-alternating if each partial sum $s_1 + \cdots + s_r$, $r \in \langle k \rangle$, is either 0 or 1. For example, the rows and columns of an alternating sign matrix are sign-alternating sequences with sum 1. Let $\mathscr{B}'_n$ be the set of $(2n+1) \times 2n$ matrices $A = [a_{ij}]$ such that:

A is invariant under 180° rotation; i.e., $a_{ij} = a_{2n+2-i,\, 2n+1-j}$ for all appropriate i and j.

Each column of A is sign-alternating with sum 1.

Each row of A, except the $(n+1)$th, is sign-alternating with sum 1.

The sequence $(a_{n+1,1}, ..., a_{n+1,n})$ is sign-alternating with sum 0.

We remark that a matrix A is in $\mathscr{B}'_n$ if and only if its transpose ${}^tA$ is in the set $\mathscr{C}'_n$ that appears in Theorem 4.4 of [O].

Let $L = L(n) = \{(i, j; k, l)\colon 1 \leqslant i < k \leqslant 2n+1,\ 2n \geqslant j > l \geqslant 1\}$ and define $i(A) = \sum_{(i,\, j;\, k,\, l) \in L} a_{ij} a_{kl}$ for $A \in \mathscr{B}'_n$. Define the following subsets of L:

0097-3165/97 $25.00

$$L_0 = \{(i, j; k, l) \in L : i = n+1 \text{ or } k = n+1\},$$
$$L_1 = \{(i, j; k, l) \in L : i+k = 2n+2,\ j+l = 2n+1\},$$
$$L_2 = L \backslash (L_0 \cup L_1),$$
$$L_+ = \{(i, j; k, l) \in L : k \leqslant n\},$$
$$L_\pm = \{(i, j; k, l) \in L : i \leqslant n,\ k \geqslant n+2\}.$$

For $* \in \{0, 1, 2, +, \pm\}$ and $A \in \mathscr{B}'_n$, put $i_*(A) = \sum_{(i, j; k, l) \in L_*} a_{ij} a_{kl}$. Define $i_1^+(A) = \#\{(i, j):\ i \leqslant n,\ j \geqslant n+1,\ a_{ij} = 1\}$ and $i_1^-(A) = \#\{(i, j):\ i \leqslant n,\ j \geqslant n+1,\ a_{ij} = -1\}$. Let $s(A)$ denote the number of -1's in A.

For each $i \in \langle n \rangle$, let ε_i be the vector (written as a $2n \times 1$ matrix) with a 1 as its ith component, a -1 as its $(2n+1-i)$th component, and 0's elsewhere. Given $\alpha = \sum_i \alpha_i \varepsilon_i$, we write $x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}$, where $x_1, x_2, ..., x_n$ are commuting indeterminates.

Finally, let

$$\delta'(B_n) = (n - \tfrac{1}{2}, ..., \tfrac{3}{2}, \tfrac{1}{2}, 0, -\tfrac{1}{2}, -\tfrac{3}{2}, ..., -n + \tfrac{1}{2})$$

and

$$\delta(B_n) = (n - \tfrac{1}{2}, ..., \tfrac{3}{2}, \tfrac{1}{2}, -\tfrac{1}{2}, -\tfrac{3}{2}, ..., -n + \tfrac{1}{2}),$$

these vectors being written as $(2n+1) \times 1$ and $2n \times 1$ matrices, respectively. We can now state our deformation of Weyl's formula.

THEOREM 1. *For $n \geqslant 1$, we have*

$$\prod_{1 \leqslant i \leqslant n} (1 + t x_i) \prod_{1 \leqslant i < j \leqslant n} (1 + t x_i x_j)(1 + t x_i x_j^{-1}) = \sum_{A \in \mathscr{B}'_n} b_A(t)\, x^{\delta'(B_n) - A\, \delta(B_n)},$$

where $b_A(t) = t^{i_1^+(A) + (i_0(A) + i_2(A))/2} (1 + 1/t)^{s(A)/2}$.

When $t = -1$, Theorem 1 gives us Weyl's formula for the root system B_n. In this case, the only matrices A to appear in the sum are those with $s(A) = 0$. Such matrices have only 0's in their $(n+1)$th rows; crossing out these rows, we obtain a set of $2n \times 2n$ matrices. These form the Weyl group of $B_n = \{\pm\varepsilon_i,\ 1 \leqslant i \leqslant n;\ \pm\varepsilon_i \pm \varepsilon_j,\ 1 \leqslant i < j \leqslant n\}$.

MONOTONE TRIANGLES, PARTITIONS, AND $\mathscr{B}'_n$

Let $\lambda = (\lambda_1, ..., \lambda_n)$ be a partition, with $n \geqslant \lambda_1 \geqslant \cdots \geqslant \lambda_n \geqslant 0$. Let $\delta_n = (n, ..., 2, 1)$. We denote by $\mathscr{M}(\lambda + \delta_n)$ the set of monotone triangles

$$T=\begin{matrix} t_{11} & & & \\ t_{21} & t_{22} & & \\ \vdots & \vdots & \ddots & \\ t_{n1} & t_{n2} & \cdots & t_{nn} \end{matrix}, \qquad t_{ij} \leqslant t_{i-1,\,j} \leqslant t_{i,\,j+1}, \quad t_{ij} < t_{i,\,j+1},$$

whose nth rows contain $\lambda_n + 1, \lambda_{n-1} + 2, ..., \lambda_1 + n$.

For any matrix $A = [a_{ij}]$, let $\hat{A} = [\hat{a}_{ij}]$ be the "column sum matrix" of A: $\hat{a}_{ij} = \sum_{k=1}^{i} a_{kj}$. The correspondence $A \mapsto \hat{A}$ is a bijection on matrices of any particular size. Now given $T \in \mathscr{M}(\lambda + \delta_n)$, let $A = A(T)$ be the $(2n+1) \times 2n$ matrix $[a_{ij}]$ with the following properties:

$$\text{For each } i \in \langle n \rangle,\ \hat{a}_{ij} = \begin{cases} 1, & \text{if } \ j = t_{il} \text{ for some } l; \\ 0, & \text{otherwise.} \end{cases}$$

$$\text{For } n+2 \leqslant i \leqslant 2n+1,\ a_{ij} = a_{2n+2-i,\, 2n+1-j}.$$

$$\text{For } j \in \langle 2n \rangle,\ a_{n+1,\,j} = 1 - \sum_{i=1}^{n} a_{ij} - \sum_{i=1}^{n} a_{i,\,2n+1-j} = a_{n+1,\,2n+1-j}.$$

Clearly A is invariant under $180°$ rotation. It is also easy to see that all columns of A are sign-alternating with sum 1, as are all rows of A other than the $(n+1)$th row. Meanwhile, the sequence $(a_{n+1,\,1}, ..., a_{n+1,\,n})$ is made up of 1's, 0's, and -1's, and its sum is 0.

Let $P_{-1,\,0}(n)$ denote the set of partitions λ whose Frobenius representations $(\alpha \mid \beta)$ satisfy

$$n-1 \geqslant \beta_1 \geqslant \alpha_1 \geqslant \beta_2 \geqslant \alpha_2 \geqslant \cdots \geqslant \beta_{p(\lambda)} \geqslant \alpha_{p(\lambda)}.$$

Then we have:

LEMMA 2. *Let λ be a partition of at most n parts, each of which is no larger than n. Let $T \in \mathscr{M}(\lambda + \delta_n)$ and let $A = A(T)$. Then the sequence $(a_{n+1,\,1}, ..., a_{n+1,\,n})$ is sign-alternating if and only if $\lambda \in P_{-1,\,0}(n)$.*

Proof. Observe that the sequence $(\hat{a}_{n1}, ..., \hat{a}_{n,\,2n})$ depends only upon λ. For $i, j \in \{0, 1\}$ and $r \in \langle n \rangle$, define $\#_{ij}(r)$ to be the number of $k \in \langle r \rangle$ such that $\hat{a}_{nk} = i$ and $\hat{a}_{n,\,2n+1-k} = j$. We have:

$$\#_{00}(r) + \#_{01}(r) + \#_{10}(r) + \#_{11}(r) = r;$$

$$\#_{00}(r) \text{ is the number of 1's in the set } \{a_{n+1,\,1}, ..., a_{n+1,\,r}\};$$

$$\#_{11}(r) \text{ is the number of } -1\text{'s in the set } \{a_{n+1,\,1}, ..., a_{n+1,\,r}\}.$$

We conclude that

$$\sum_{k=1}^{r} a_{n+1,k} = \#_{00}(r) - \#_{11}(r)$$

$$= (\#_{00}(r) + \#_{01}(r)) - (\#_{01}(r) + \#_{11}(r))$$

$$= \#\{k \in \langle r \rangle : \hat{a}_{nk} = 0\} - \#\{k \in \langle r \rangle : \hat{a}_{n,\,2n+1-k} = 1\}.$$

We now recall a simple, but useful, lemma, due to Macdonald.

LEMMA 3 [M, Chap. I, (1.7)]. *Let λ be a partition with at most n parts, each of which is no larger than m. Then the sets $\{\lambda_i + n - i + 1 : i \in \langle n \rangle\}$ and $\{n + j - \lambda'_j : j \in \langle m \rangle\}$ are disjoint, and their union is $\langle n + m \rangle$.*

Let $(\alpha \mid \beta)$ be the Frobenius representation of λ. Lemma 3 implies that $\hat{a}_{nk} = 0$ if and only if $2n + 1 - k = \lambda'_j + n - j + 1$ for some $j \in \langle n \rangle$. If $k \leqslant r$, then $\lambda'_j - j = n - k \geqslant n - r \geqslant 0$, so $\lambda'_j - j = \beta_j$. So $\#_{00}(r) + \#_{01}(r)$ is the number of j such that $\beta_j \geqslant n - r$, equivalently the largest j such that $\beta_j \geqslant n - r$. Similarly, $\hat{a}_{n,\,2n+1-k} = 1$ if and only if $2n + 1 - k = \lambda_i + n - i + 1$ for some $i \in \langle n \rangle$, and we see that $\#_{01}(r) + \#_{11}(r)$ is the largest i such that $\alpha_i \geqslant n - r$.

The preceding remarks imply that $\#_{00}(r) - \#_{11}(r) \geqslant 0$ for all $r \in \langle n \rangle$ if and only if $\beta_j \geqslant \alpha_j$ for all $j \leqslant p(\lambda)$, and that $\#_{00}(r) - \#_{11}(r) \leqslant 1$ for all $r \in \langle n \rangle$ if and only if $\alpha_i \geqslant \beta_{i+1}$ for all $i \leqslant p(\lambda) - 1$. Thus the sequence $(a_{n+1,1}, ..., a_{n+1,n})$ is sign-alternating if and only if $\beta_1 \geqslant \alpha_1 \geqslant \beta_2 \geqslant \alpha_2 \geqslant \cdots \geqslant \beta_{p(\lambda)} \geqslant \alpha_{p(\lambda)}$, and this completes the proof of Lemma 2.

As a corollary of Lemma 2, we see that $T \mapsto A(T)$ defines a bijection of $\bigcup_\lambda \mathscr{M}(\lambda + \delta_n)$, $\lambda \in P_{-1,0}(n)$, onto $\mathscr{B}'_n$.

Recall the functions $i(A)$ and $i_*(A)$, $* \in \{0, 1, 2, +, \pm\}$, defined on matrices $A \in \mathscr{B}'_n$. It is easy to see that $i(A) = i_0(A) + i_1(A) + i_2(A)$, $i_1(A) + i_2(A) = 2i_+(A) + i_\pm(A)$, and $i_1(A) = i_1{}^+(A) + i_1{}^-(A)$ for any $A \in \mathscr{B}'_n$.

If T is any monotone triangle, then we define

$$\max(T) = \#\{(i, j) : t_{ij} < t_{i-1,\,j} = t_{i,\,j+1}\};$$

$$\mathrm{sp}(T) = \#\{(i, j) : t_{ij} < t_{i-1,\,j} < t_{i,\,j+1}\};$$

$$x^T = x_1^{s_1} x_2^{s_2 - s_1} \cdots x_n^{s_n - s_{n-1}}, \qquad \text{where} \quad s_i = \sum_{l=1}^{i} t_{il}.$$

And if λ is any partition, we put

$$t(\lambda) = \#\{(i, j) : i \leqslant j,\, \lambda_i + \lambda_{j+1} > i + j\};$$

$$v(\lambda) = \#\{(i, j) : i \leqslant j,\, \lambda_i + \lambda_{j+1} = i + j\}.$$

The following lemma tells us how functions on $A \in \mathscr{B}'_n$ are related to functions on the partitions and monotone triangles corresponding to A.

LEMMA 4. *Let* $\lambda \in P_{-1,0}(n)$; *let* $T \in \mathscr{M}(\lambda + \delta_n)$ *and* $A = A(T)$. *Then*:

(1) $\max(T) + \mathrm{sp}(T) = i_+(A)$;

(2) $v(\lambda) + \mathrm{sp}(T) = \frac{1}{2}s(A)$;

(3) $p(\lambda) = i_1^+(A) - i_1^-(A)$;

(4) $2t(\lambda) + p(\lambda) = i_\pm(A)$;

(5) $|\lambda| - (2t(\lambda) + p(\lambda)) = \frac{1}{2}i_0(A)$;

(6) $x^T x_1^{-1} x_2^{-2} \cdots x_n^{-n} = x^{\delta'(B_n) - A\,\delta(B_n)}$.

Proof. (1) follows from Proposition 1.1 of [O].

To prove (2), we observe that $\mathrm{sp}(T)$ is the number of -1's in the first n rows of A (this is also contained in Proposition 1.1 of [O]). So if $s_0(A)$ is the number of -1's in the $(n+1)$th row of A, we see that $\mathrm{sp}(T) = (s(A) - s_0(A))/2$. We must therefore show that $s_0(A) = 2v(\lambda)$, or equivalently that $v(\lambda)$ is the number of -1's in the sequence $(a_{n+1,1}, ..., a_{n+1,n})$. Now if $k \in \langle n \rangle$ and $a_{n+1,k} = -1$, we have $\hat{a}_{nk} = \hat{a}_{n,2n+1-k} = 1$; there is a pair (i, j) with $i \leqslant j$, $k = \lambda_{j+1} + n - j$, and $2n+1-k = \lambda_i + n - i + 1$, and we see that $\lambda_i + \lambda_{j+1} = i + j$. Conversely, if $\lambda_i + \lambda_{j+1} = i + j$, then we must have $i \leqslant p(\lambda)$ and $j > p(\lambda)$; therefore $\lambda_i + n - i + 1 \geqslant n + 1$, $\lambda_{j+1} + n - j \leqslant n$, and $\lambda_i + n - i + 1 + \lambda_{j+1} + n - j = 2n + 1$. Then if $k = \lambda_{j+1} + n - j$, we have $k \in \langle n \rangle$ and $\hat{a}_{nk} = \hat{a}_{n,2n+1-k} = 1$, so $a_{n+1,k} = -1$. This completes the proof of (2).

(3) is easy; $i_1^+(A) - i_1^-(A) = \sum_{j \geqslant n+1, i \leqslant n} a_{ij} = \sum_{j \geqslant n+1} \hat{a}_{nj} = \#\{j : \lambda_j + n - j + 1 \geqslant n + 1\} = p(\lambda)$.

To prove (4), observe that

$$i_\pm(A) = \sum_{1 \leqslant j < k \leqslant 2n} \left(\hat{a}_{nk} \sum_{l=n+2}^{2n+1} a_{lj} \right) = \sum_{1 \leqslant j < k \leqslant 2n} \hat{a}_{nk} \hat{a}_{n,2n+1-j},$$

where the latter equality is a consequence of the rotational symmetry of A. The summand on the right is 0, unless both k and $2n+1-j$ belong to the set $\{\lambda_i + n - i + 1 : i \in \langle n \rangle\}$, in which case it is 1. Now if $k = \lambda_l + n - l + 1$ and $2n + 1 - j = \lambda_m + n - m + 1$, then $\lambda_l + \lambda_m = l + m - 1 + k - j > l + m - 1$, so we find that

$$\begin{aligned} i_\pm(A) &= \#\{(l, m) : \lambda_l + \lambda_m > l + m - 1\} \\ &= 2\#\{(l, m) : l < m, \lambda_l + \lambda_m > l + m - 1\} \\ &\quad + \#\{(l, m) : l = m, \lambda_l + \lambda_m > l + m - 1\} \\ &= 2t(\lambda) + p(\lambda). \end{aligned}$$

For (5), we observe that the rotational symmetry of A implies that $\hat{a}_{ni}+\hat{a}_{n+1,2n+1-i}=1$ for all $i\in\langle 2n\rangle$. This tells us that $\hat{a}_{n+1,i}$ is 0 or 1, according as i is or is not in the set $\{n+k-\lambda_k\colon k\in\langle n\rangle\}$. Again using rotational symmetry, we see that $\frac{1}{2}i_0(A)=\sum_{1\leqslant j<k\leqslant 2n} a_{n+1,j}\hat{a}_{nk}$. Since $\hat{a}_{nk}$ is 1 exactly when $k=\lambda_l+n-l+1$ for some $l\in\langle n\rangle$, and $a_{n+1,j}=\hat{a}_{n+1,j}-\hat{a}_{nj}$, we can rewrite the latter sum as $\sum_{l=1}^{n}\sum_{j=1}^{\lambda_l+n-l}(\hat{a}_{n+1,j}-\hat{a}_{nj})$. Now

$$\begin{aligned}\sum_{l=1}^{n}\sum_{j=1}^{\lambda_l+n-l}\hat{a}_{n+1,j}&=\sum_{l=1}^{n}(\lambda_l+n-l-\#\{k\colon n+k-\lambda_k<\lambda_l+n-l+1\})\\&=\sum_{l=1}^{n}(\lambda_l+n-l-\#\{k\colon \lambda_k+\lambda_l>k+l-1\})\end{aligned}$$

and

$$\begin{aligned}\sum_{l=1}^{n}\sum_{j=1}^{\lambda_l+n-l}\hat{a}_{nj}&=\sum_{l=1}^{n}\#\{k\colon \lambda_k+n-k+1<\lambda_l+n-l+1\}\\&=\sum_{l=1}^{n}(n-l).\end{aligned}$$

So

$$\begin{aligned}\tfrac{1}{2}i_0(A)&=\sum_{l=1}^{n}(\lambda_l-\#\{k\colon \lambda_k+\lambda_l>k+l-1\})\\&=|\lambda|-\#\{(k,l)\colon \lambda_k+\lambda_l>k+l-1\}\\&=|\lambda|-(2t(\lambda)+p(\lambda)).\end{aligned}$$

(6) is left as an easy exercise for the reader.

PROOF OF THE IDENTITY

We need two more results to prove Theorem 1.

PROPOSITION 5 [O, remark following proof of Lemma 3.5]. *For $n\geqslant 1$, we have*

$$\prod_{1\leqslant i\leqslant n}(1+tx_i)\prod_{1\leqslant i<j\leqslant n}(1+x_ix_j)=\sum_{\lambda\in P_{-1,0}(n)}t^{|\lambda|-2t(\lambda)}\left(1+\frac{1}{t^2}\right)^{v(\lambda)}s_\lambda(x_1,\ldots,x_n).$$

PROPOSITION 6 [O, Corollary 1.3]. *Let λ be a partition with length $\leqslant n$. Then we have*

$$s_\lambda(x_1, \ldots, x_n) \prod_{1 \leqslant i < j \leqslant n} (1 + t x_i x_j^{-1})$$

$$= \sum_{T \in \mathscr{M}(\lambda + \delta_n)} t^{\max(T) + \operatorname{sp}(T)} \left(1 + \frac{1}{t}\right)^{\operatorname{sp}(T)} x^T x_1^{-1} x_2^{-2} \cdots x_n^{-n}.$$

Proposition 5 is equivalent to one of the two identities given in Theorem 3.1.2 of [S]; it is not proved in [O]. Proposition 6 is just a restatement of Theorem 2.1 of [T].

Proof of Theorem 1. Beginning with Proposition 5, replace t with $t^{1/2}$ and replace each x_i with $t^{1/2} x_i$:

$$\prod_{1 \leqslant i \leqslant n} (1 + t x_i) \prod_{1 \leqslant i < j \leqslant n} (1 + t x_i x_j)$$

$$= \sum_{\lambda \in P_{-1,0}(n)} t^{|\lambda| - t(\lambda)} \left(1 + \frac{1}{t}\right)^{v(\lambda)} s_\lambda(x_1, \ldots, x_n).$$

Multiply both sides by $\prod_{1 \leqslant i < j \leqslant n} (1 + t x_i x_j^{-1})$, and apply Proposition 6:

$$\prod_{1 \leqslant i \leqslant n} (1 + t x_i) \prod_{1 \leqslant i < j \leqslant n} (1 + t x_i x_j)(1 + t x_i x_j^{-1})$$

$$= \sum_{\substack{\lambda \in P_{-1,0}(n) \\ T \in \mathscr{M}(\lambda + \delta_n)}} t^{|\lambda| - t(\lambda) + \max(T) + \operatorname{sp}(T)} \left(1 + \frac{1}{t}\right)^{v(\lambda) + \operatorname{sp}(T)} x^T x_1^{-1} \cdots x_n^{-n}.$$

Now we employ Lemmas 2 and 4. Under the correspondence $T \mapsto A(T)$, we have $v(\lambda) + \operatorname{sp}(T) = \frac{1}{2} s(A)$; $x^T x_1^{-1} \cdots x_n^{-n} = x^{\delta'(B_n) - A\delta(B_n)}$; and

$$|\lambda| - t(\lambda) + \max(T) + \operatorname{sp}(T) = \tfrac{1}{2}(i_0(A) + i_\pm(A) + i_1^+(A) - i_1^-(A) + 2i_+(A))$$

$$= \tfrac{1}{2}(i(A) + 2i_1^+(A) - i_1(A))$$

$$= i_1^+(A) + \frac{i_0(A) + i_2(A)}{2}.$$

This completes the proof.

REFERENCES

[M] I. G. Macdonald, "Symmetric Functions and Hall Polynomials," Clarendon, Oxford, 1979.

[O] S. Okada, Alternating sign matrices and some deformations of Weyl's denominator formula, *J. Alg. Combin.* **2** (1993), 155–176.

[S] T. Simpson, "Combinatorial Proofs and Generalizations of Weyl's Denominator Formula," Ph.D. thesis, Pennsylvania State University, 1994.

[T] T. Tokuyama, A generating function of strict Gelfand patterns and some formulas on characters of general linear groups, *J. Math. Soc. Japan* **40** (1988), 671–685.